M.U
M.U.C. 93

QNo→ State and Prove Laurent's theorem.

Ans. Statement:- Let $f(z)$ be analytic in the ring Shaped region D bounded by two Concentric Circles C_1 & C_2 With centre z_0 and radii ρ_1 & ρ_2 $(\rho_1 > \rho_2)$ and let z be any Point of D. Then,

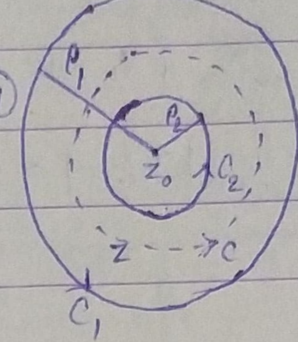
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

Where, $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z_1)}{(z_1-z_0)^{n+1}} dz_1$

and $b_n = \frac{1}{2\pi i}\int_{C_2} (z_1-z_0)^{n-1} f(z_1)\, dz_1$.

Proof:- Let z be any Point of the ring-shaped region D. Then by Cauchy's integral formula for doubly Connected region,

$$f(z) = \frac{1}{2\pi i}\left[\int_{C_1} \frac{f(z)}{\xi - z}\, d\xi - \int_{C_2} \frac{f(z)}{\xi - z}\, d\xi\right] \quad \text{--- (1)}$$



We Consider the integral

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)}{\xi - z}\, d\xi.$$

for any Point ξ on C_1, we have

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{(\xi - z_0)\left\{1 - \frac{z - z_0}{\xi - z_0}\right\}}$$

$$= \frac{1}{(\xi - z_0)}\left\{1 - \frac{z - z_0}{\xi - z_0}\right\}^{-1}$$

$$= \frac{1}{\xi - z_0}\left\{1 + \frac{z - z_0}{\xi - z_0} + \left(\frac{z - z_0}{\xi - z_0}\right)^2 + \cdots + \left(\frac{z - z_0}{\xi - z_0}\right)^{n-1} + \left(\frac{z - z_0}{\xi - z_0}\right)^n \cdot \frac{1}{1 - \frac{z - z_0}{\xi - z_0}}\right]$$

$$= \frac{1}{\xi - z_0} + \frac{z - z_0}{(\xi - z_0)^2} + \frac{(z - z_0)^2}{(\xi - z_0)^3} + \cdots + \frac{(z - z_0)^{n-1}}{(\xi - z_0)^n} + \frac{(z - z_0)^n}{(\xi - z_0)^n(\xi - z)}$$

$$\therefore \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)}{\xi - z}\, d\xi = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)}{\xi - z_0}\, d\xi + \frac{(z - z_0)}{2\pi i}\int_{C_1} \frac{f(\xi)}{(\xi - z_0)^2}\, d\xi$$

$$+ \cdots + \frac{(z - z_0)^{n-1}}{2\pi i}\int_{C_1} \frac{f(\xi)}{(\xi - z_0)^n}\, d\xi + R_n \quad \text{--- (2)}$$

Where $R_n = \frac{(z_0 - z_0)^n}{2\pi i} \int_{C_1} \frac{f(\xi)\, d\xi}{(\xi - z_0)^n (\xi - z)}$

Let $|z - z_0| = r$, where $\rho_2 < r < \rho_1$.

Also, $|\xi - z_0| = \rho_1$

$\therefore |\xi - z| = |(\xi - z_0) - (z - z_0)| \geq |\xi - z_0| - |z - z_0| = \rho_1 - r.$

$$\therefore |R_n| = \left| \frac{(z - z_0)^n}{2\pi i} \int_{C_1} \frac{f(\xi)\, d\xi}{(\xi - z_0)^n (\xi - z)} \right| \leq \frac{|z - z_0|^n}{2\pi} \int_{C_1} \frac{|f(\xi)||d\xi|}{|\xi - z_0|^n |\xi - z|}$$

$$= \frac{M r^n}{2\pi (\rho_1 - r) \rho_1^n}$$

$$\int_{C_1} |d\xi| = \frac{M r^n \cdot \cancel{2\pi} \rho_1}{\cancel{2\pi} (\rho_1 - r) \rho_1^n} = \frac{M \rho_1}{\rho_1 - r} \left(\frac{r}{\rho_1} \right)^n \to 0 \text{ as } n \to \infty$$

since, $\frac{r}{\rho_1} < 1$.

Where M is maximum value of $f(\xi)$ on C_1.

$$\therefore \frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)}{\xi - z}\, d\xi = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Again, $$-\frac{1}{\xi - z} = \frac{1}{(z - z_0)(\xi - z_0)} = \frac{1}{(z - z_0)\left\{ 1 - \frac{\xi - z_0}{z - z_0} \right\}}$$

$$= \frac{1}{(z - z_0)} \left(1 - \frac{\xi - z_0}{z - z_0} \right)^{-1}$$

$$= \frac{1}{(z - z_0)} \left[1 + \left(\frac{\xi - z_0}{z - z_0} \right) + \left(\frac{\xi - z_0}{z - z_0} \right)^2 + \cdots \left(\frac{\xi - z_0}{z - z_0} \right)^{n-1} + \left(\frac{\xi - z_0}{z - z_0} \right)^n \frac{1}{1 - \frac{\xi - z_0}{z - z_0}} \right]$$

$$= \left[\frac{1}{(z-z_0)} + \frac{(q_1 - z_0)}{(z-z_0)^2} + \frac{(q_1 - z_0)^2}{(z-z_0)^3} + \cdots + \frac{(q_1 - z_0)^{n-1}}{(z-z_0)^n} \right.$$

$$\left. + \frac{(q_1 - z_0)^n}{(z-z_0)^n (z - q_1)} \right]$$

$$\therefore -\frac{1}{2\pi i} \int_{C_2} \frac{f(q_1)\, dq_1}{q_1 - z} = \frac{1}{2\pi i} \int_{C_2} \frac{f(q_1)\, dq_1}{z - z_0} + \frac{1}{2\pi i} \int_{C_2} \frac{(q_1 - z_0) f(q_1)\, dq_1}{(z-z_0)^2}$$

$$+ \frac{1}{2\pi i} \int_{C_2} \frac{(q_1 - z_0)^2 f(q_1)}{(z - z_0)^3}\, dq_1 + \cdots + \frac{1}{2\pi i} \int_{C_2} \frac{(q_1 - z_0)^{n-1} f(q_1)\, dq_1}{(z-z_0)^n}$$

$$+ S_n$$

where, $S_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_2} \frac{(q_1 - z_0)^n f(q_1)}{(z-z_0)^n (z - q_1)}\, dq_1$.

Let M' be the maximum value of $f(q)$ on C_2

$$\text{Now, } |S_n| = \left| \frac{1}{2\pi i} \int_{C_2} \frac{(q_1 - z_0)^n f(q_1)}{(z-z_0)^n (z - q_1)}\, dq_1 \right|$$

$$\leq \frac{1}{2\pi |z - z_0|^n} \int_{C_2} \frac{|q_1 - z_0|^n |f(q_1)|}{|z - q_1|} |dq_1|$$

Since, $|z - z_0| = r$, $|q_1 - z_0| = \rho_2$, we have

$$|z - q_1| = |(z - z_0) - (q_1 - z_0)| \geq |z - z_0| - |q_1 - z_0| = r - \rho_2.$$

We have

$$|S_n| \leq \frac{M' \rho_2^n}{2\pi r^n (r - \rho_2)} \int_{C_2} |dq_1|$$

$$= \frac{M' \cdot \cancel{2\pi} \rho_2}{\cancel{2\pi} (r - \rho_2)} \cdot \left(\frac{\rho_2}{r}\right)^n \to 0 \text{ as } n \to \infty$$

$$\therefore -\frac{1}{2\pi i}\int_{C_2}\frac{f(q_1)}{q_1-z}\,dq_1=\sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$$

Hence, $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n+\sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$.

QNo→ State and Prove the uniqueness theorem

Ans→ Statement:- Suppose that we have obtained in any manner or as the definition of f(z) the formula,

$$f(z)=\sum_{n=-\infty}^{\infty} P_n(z-z_0)^n \quad (\rho_2<|z-z_0|<\rho_1).$$

then the series is necessarily identical with the Laurent series.

Proof:- Let the circle C be $|z-z_0|=\rho$, where $\rho_2<\rho<\rho_1$. Then the coefficient a_n of $(z-z_0)^n$ is the Laurent's expansion will be given by,

$$a_n=\frac{1}{2\pi i}\int_C\frac{f(q_1)\,dq_1}{(q_1-z_0)^{n+1}}=\frac{1}{2\pi i}\int_C\frac{\sum_{m=-\infty}^{\infty} P_m(q_1-z_0)^m}{(q_1-z_0)^{n+1}}\,dq_1$$

$$=\frac{1}{2\pi i}\sum_{m=-\infty}^{\infty} P_m\int_C\frac{(q_1-z_0)^m}{(q_1-z_0)^{n+1}}\,dq_1$$

Which is Justified since the series under integration is uniformly convergent

on the every closed subset of the annulus,
Let $q - z_0 = \rho e^{i\theta}$

$\therefore dq = \rho i e^{i\theta} d\theta$

$$\therefore a_n = \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty} P_m \int_0^{2\pi} \frac{\rho^m \cdot e^{mi\theta}}{\rho^{n+1} \cdot e^{(n+1)i\theta}} \rho i e^{i\theta} d\theta$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} P_m \int_0^{2\pi} \rho^{m-n} e^{(m-n)i\theta} d\theta.$$

$$= \int_0^{2\pi} e^{i(m-n)\theta} d\theta = \left\{ \frac{e^{i(m-n)\theta}}{i(m-n)} \right\}_0^{2\pi} = 0 \text{ if } m \neq n,$$

and $m = n$

$$\int_0^{2\pi} e^{i(m-n)\theta} d\theta = \int_0^{2\pi} d\theta = (\theta)_0^{2\pi} = 2\pi$$

$\therefore a_n = P_n$

Hence $f(z) = \sum_{n=-\infty}^{\infty} P_n (z - z_0)^n = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$

Therfore, the given series is indentical of the Laurent's extension of $f(z)$.